QUANTUM TOROIDAL ALGEBRAS AND THEIR VERTEX REPRESENTATIONS

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Abstract. We construct the vertex representations of the quantum toroidal algebras $U_q(\mathfrak{sl}_{n+1,tor})$. In the classical case the vertex representations are not irreducible. However in the quantum case they are irreducible.
For n=1, we construct a set of finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$.

1. Introduction

The classical toroidal algebras have been studied by many authors [MEY], [S], [Y], etc. Here "classical" means $q = 1$. The definition of the quantum toroidal algebras is given in [GKV]. They gave a geometric realization of the quantum toroidal algebras without any results on their representation theory. Recently Varagnolo and Vasserot [VV] proved a Schur-type duality between representations of the quantum toroidal algebras and the double affine Hecke algebra introduced by Cherednik [C]. This is an analogue of the duality between the quantum affine algebras and the affine Hecke algebras given by Chari and Pressly [CP]. In [VV] only the representations of "trivial central charge" was studied. It is known that there are two subalgebras $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ and $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$ of $U_q(\mathfrak{sl}_{n+1,tor})$ which are isomorphic to $U_q(\widehat{\mathfrak{sl}}_{n+1})$. We say the $U_q(\mathfrak{sl}_{n+1,tor})$-module M has the trivial central charge if M has level 0 as both $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module and $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$-module. It is an analogue of level 0 representations of the affine quantum algebras. In this paper we say that M has a level $(0,0)$ instead of the trivial central charge. The first 0 means that M has a level 0 as a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module and the second 0 means that M has a level 0 as a $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$-module.

In this paper we try to consider an analogue of "integrable representations" in the toroidal case. Let us recall the integrability of quantum Kac-Moody modules. Let $U_q(\mathfrak{g})$ be a quantum Kac-Moody algebra and V a $U_q(\mathfrak{g})$-module. We say V is integrable if V has a weight space decomposition and locally nilpotent actions of the Chevalley generators of

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$U_q(\mathfrak{g})$. Therefore the definition of "integrability" needs Chevalley-type generators. The toroidal algebras is defined through Drinfeld type of generators and its Chevalley-type generators are not known. Therefore we are not able to define the integrability at this moment. However, in the affine case, Frenkel-Jing [FJ] realized the integrable representations with level 1 by the vertex representations. Thus if there are "vertex representations" of quantum toroidal algebras, they must be interesting examples of the integrable representations still not defined. In the $q = 1$ case, vertex representations of the toroidal algebras have been already considered by Moody-Eswara Rao-Yokonuma [MEY]. In this paper for we construct the q-analogue of the representations defined by them for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ with level $(1,0)$ and $(1,1)$. Therefore we give a new class of the representations of the quantum toroidal algebras. In the $q = 1$ case, the Fock modules are not irreducible over the Heisenberg algebra and the vertex representations are not irreducible over the toroidal algebra. In the quantum case, it is not the case: the Fock modules are irreducible and also the vertex representations with level $(1,0)$ are irreducible.

The algebra $U_q(\mathfrak{g}_{tor})$ has infinitely many generators satisfying infinitely many relations (See §2). It is preferable that $U_q(\mathfrak{g}_{tor})$ is written by finitely many generators with finitely many relations. According to [GKV] there are finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$ but the relations among these generators are highly non-trivial. In this paper we give an explicit form of finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$ and closed relations of them (See §4). They coincide with the generators by Vasserot [V].

2. Definition of quantum toroidal algebras

2.1. **Notations.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra of type A_n and $\widehat{\mathfrak{g}}$ an affine Kac-Moody Lie algebra of type $A_n^{(1)}$. We denote their Cartan subalgebras by $\mathfrak{h}$ and $\widehat{\mathfrak{h}}$ respectively . We denote by $\overline{\alpha}_1, \cdots, \overline{\alpha}_n$ the simple roots of $\mathfrak{g}$, by $\overline{h}_1, \cdots, \overline{h}_n$ the simple coroots of $\mathfrak{g}$, by $\overline{\Lambda}_1, \cdots, \overline{\Lambda}_n$ the fundamental weights of $\mathfrak{g}$, by $\alpha_0, \cdots, \alpha_n$ the simple roots of $\widehat{\mathfrak{g}}$, by $h_0, \cdots, h_n$ the simple coroots of $\widehat{\mathfrak{g}}$ and $\Lambda_0, \cdots, \Lambda_n$ the fundamental weights of $\widehat{\mathfrak{g}}$. Let $\overline{Q} = \oplus_{i=1}^n \mathbb{Z}\overline{\alpha}_i$ be the root lattice of $\mathfrak{g}$, $\overline{P} = \oplus_{i=1}^n \mathbb{Z}\overline{\Lambda}_i$ the weight lattice of $\mathfrak{g}$, $Q = \oplus_{i=0}^n \mathbb{Z}\alpha_i$ the root lattice of $\widehat{\mathfrak{g}}$ and $P = \oplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ the weight lattice of $\widehat{\mathfrak{g}}$. Here δ is the null root.

We denote the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$ (*resp.* $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$) by $\langle\ ,\ \rangle$. The invariant bilinear form on P is given by $(\alpha_i|\alpha_j) = -\delta_{ij-1} + 2\delta_{ij} - \delta_{ij+1}$

and $(\delta|\delta) = 0$. The projection form P to $\overline{P}$ is given by $\overline{\Lambda_i} = \Lambda_i - \Lambda_0$ and $\overline{\delta} = 0$.

2.2. We will give the definition of the quantum toroidal algebra $U_q(\mathfrak{g}_{tor})$.

Definition 2.2.1. *Let $M = (m_{ij})_{0 \le i,j \le n}$ be a skew-symmetric $(n+1) \times (n+1)$-matrix with integral coefficients and let κ be an element of $\mathbb{Q}(q)^{\times}$. $U_q(\mathfrak{g}_{tor})$ is an associated algebra over $\mathbb{Q}(q)$ with generators :*

$$E_{i,k},\ F_{i,k},\ H_{i,l},\ K_i^{\pm},\ q^{\pm\frac{1}{2}c}, q^{\pm d_1}, q^{\pm d_2},$$

for $k \in \mathbb{Z}$, $l \in \mathbb{Z}\backslash\{0\}$ and $i = 0, 1, \cdots, n$.

We introduce $K_{i,k}^{\pm}$ as the Fourier components of the following generating functions:

$$K_i^+(z) = \sum_{k \ge 0} K_{i,k}^+ z^{-k} = K_i^+ \exp((q-q^{-1}) \sum_{k \ge 1} H_{i,k} z^{-k}),$$

$$K_i^-(z) = \sum_{k \le 0} K_{i,k}^- z^{-k} = K_i^- \exp(-(q-q^{-1}) \sum_{k \ge 1} H_{i,-k} z^{k}).$$

The defining relations of $U_q(\mathfrak{g}_{tor})$ are then written as follows:

$$q^{\pm\frac{1}{2}c} \text{ are central,} \tag{2.2.1}$$

$$K_i^+ K_i^- = K_i^- K_i^+ = 1, \tag{2.2.2}$$

$$[K_i^{\pm}, K_j^{\pm}] = 0, \tag{2.2.3}$$

$$[K_i^{\pm}, H_{j,l}] = 0, \tag{2.2.4}$$

$$[H_{i,k}, H_{j,l}] = \delta_{k+l,0} \frac{1}{k} [k\langle h_i, \alpha_j \rangle] \frac{q^{kc} - q^{-kc}}{q - q^{-1}} \kappa^{-km_{ij}}, \tag{2.2.5}$$

$$[q^{\pm d_i}, K_j^{\pm}] = 0, \tag{2.2.6}$$

$$q^{d_1} H_{j,l} q^{-d_1} = q^l H_{j,l}, \tag{2.2.7}$$

$$[q^{\pm d_2}, H_{j,l}] = 0, \tag{2.2.8}$$

$$q^{d_1} E_{j,k} q^{-d_1} = q^k E_{j,k}, \tag{2.2.9}$$

$$q^{d_1} F_{j,k} q^{-d_1} = q^k F_{j,k},$$

$$q^{d_2} E_{j,k} q^{-d_2} = q^{\delta_{j0}} E_{j,k}, \tag{2.2.10}$$

$$q^{d_2} F_{j,k} q^{-d_2} = q^{-\delta_{j0}} F_{j,k},$$

$$K_i^+ E_{j,k} K_i^- = q^{\langle h_i,\alpha_j\rangle} E_{j,k}, \tag{2.2.11}$$

$$K_i^+ F_{j,k} K_i^- = q^{-\langle h_i,\alpha_j\rangle} F_{j,k},$$

$$[H_{i,k}, E_{j,l}] = \frac{1}{k}[k\langle h_i,\alpha_j\rangle] q^{-\frac{1}{2}|k|c} \kappa^{-km_{ij}} E_{j,k+l}, \tag{2.2.12}$$

$$[H_{i,k}, F_{j,l}] = -\frac{1}{k}[k\langle h_i,\alpha_j\rangle] q^{\frac{1}{2}|k|c} \kappa^{-km_{ij}} F_{j,k+l},$$

$$\kappa^{m_{ij}} E_{i,k+1}E_{j,l} - q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}} E_{j,l}E_{i,k+1} = q^{\langle h_i,\alpha_j\rangle} E_{i,k}E_{j,l+1} - E_{j,l+1}E_{i,k}, \tag{2.2.13}$$

$$\kappa^{m_{ij}} F_{i,k+1}F_{j,l} - q^{-\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}} F_{j,l}F_{i,k+1} = q^{-\langle h_i,\alpha_j\rangle} F_{i,k}F_{j,l+1} - F_{j,l+1}F_{i,k},$$

$$[E_{i,k}, F_{j,l}] = \delta_{i,j}\frac{1}{q-q^{-1}}\{q^{\frac{1}{2}(k-l)c}K^+_{i,k+l} - q^{\frac{1}{2}(l-k)c}K^-_{i,k+l}\}, \tag{2.2.14}$$

$$\sum_{\sigma\in\mathfrak{S}_m}\sum_{r=0}^{m}(-1)^r \begin{bmatrix} m \\ r \end{bmatrix} E_{i,k_{\sigma(1)}}\cdots E_{i,k_{\sigma(r)}} E_{j,l} E_{i,k_{\sigma(r+1)}}\cdots E_{i,k_{\sigma(m)}} = 0, \tag{2.2.15}$$

$$\sum_{\sigma\in\mathfrak{S}_m}\sum_{r=0}^{m}(-1)^r \begin{bmatrix} m \\ r \end{bmatrix} F_{i,k_{\sigma(1)}}\cdots F_{i,k_{\sigma(r)}} F_{j,l} F_{i,k_{\sigma(r+1)}}\cdots F_{i,k_{\sigma(m)}} = 0,$$

for $i \neq j$,

where $m = 1 - \langle h_i,\alpha_j\rangle$.

In these relations we denote $[k] = \frac{q^k - q^{-k}}{q-q^{-1}}$, $[n]! = \prod_{k=1}^{n}[k]$, $\begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]!}{[r]![m-r]!}$.

2.3. Let $U''_q(\mathfrak{g}_{tor})$ be the subalgebra of $U_q(\mathfrak{g}_{tor})$ generated by $E_{i,k}, F_{i,k}, K_i^{\pm}, H_{i,l}$, $q^{\pm\frac{1}{2}c}$. Let $U_q^{\phi'}(\mathfrak{g}_{tor})$ (*resp* $U_q'^{\phi}(\mathfrak{g}_{tor})$) be the subalgebra generated by $U''_q(\mathfrak{g}_{tor})$ and $q^{\pm d_1}$ (*resp* $q^{\pm d_2}$). Let $U_q^{(1)'}(\widehat{\mathfrak{sl}}_{n+1})$ be the subalgebra generated by $E_{i,k}, F_{i,k}, K_i^{\pm}, H_{i,l}$, $q^{\pm\frac{1}{2}c}$ $(1 \le i \le n, k \in \mathbb{Z}, l \in \mathbb{Z}\backslash\{0\})$ and $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ the subalgebra generated by $U_q^{(1)'}(\widehat{\mathfrak{sl}}_{n+1})$ and $q^{\pm d_1}$. Let $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$ the subalgebra generated by $E_{i,0}, F_{i,0}, K_i^{\pm}$ $(0 \le i \le n)$ and $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$ the subalgebra generated by $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$ and $q^{\pm d_2}$. By the definition $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$ is isomorphic to $U_q(\widehat{\mathfrak{sl}}_{n+1})$ and $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$ is

isomorphic to $U'_q(\widehat{\mathfrak{sl}}_{n+1})$.

The following are straight forward.

Lemma 2.3.1. *For* $1 \le i \le n$ *let* $\overline{E_{i,k}} = E_{i,k}\kappa^{\sum_{j=1}^{i} km_{j-1j}}$, $\overline{F_{i,k}} = F_{i,k}\kappa^{\sum_{j=1}^{i} km_{j-1j}}$, $\overline{H_{i,l}} = H_{i,l}\kappa^{\sum_{j=1}^{i} lm_{j-1j}}$. *Then the relations between* $\overline{E_{i,k}}$, $\overline{F_{i,k}}$, $\overline{H_{i,l}}$ *and* $K_i^{\pm}$ *are precisely the relations of Drinfeld generators of* $U_q(\widehat{\mathfrak{sl}}_{n+1})$. *That is,* $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ *is isomorphic to* $U_q(\widehat{\mathfrak{sl}}_{n+1})$ *and* $U_q^{(1)\prime}(\widehat{\mathfrak{sl}}_{n+1})$ *is isomorphic to* $U'_q(\widehat{\mathfrak{sl}}_{n+1})$.

Lemma 2.3.2. *Let* $K_\delta^{\pm} = \prod_{i=0}^{n} K_i^{\pm}$. *Then* $K_\delta^{\pm}$ *are the central elements of* $U_q(\mathfrak{g}_{tor})$.

Note that $q^{\pm c}$ is the central elements of $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ and $K_\delta^{\pm}$ the central elements of $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$.

3. Vertex representations

3.1. **Heisenberg algebras.** In this section we shall give the vertex representations of $U_q(\mathfrak{g}_{tor})$. We assume $c = 1$.

Consider a $\mathbb{Q}(q)$-algebra S_n generated by $H_{i,l}$ $(0 \le i \le n,\ l \in \mathbb{Z}\backslash\{0\})$ satisfying:

$$[H_{i,k}, H_{j,l}] = \delta_{k+l,0}\frac{1}{k}[k\langle h_i, \alpha_j\rangle]\frac{q^k - q^{-k}}{q - q^{-1}}\kappa^{-km_{ij}}. \tag{3.1.1}$$

We call S_n the Heisenberg algebra.

Let S_n^+ (*resp.* S_n^-) be the subalgebra of S_n generated by $H_{i,l}$ $(0 \le i \le n,\ l > 0)$ (*resp.* $0 \le i \le n,\ l < 0$).

We introduce the Fock space

$$\mathcal{F}_n = S_n v_0$$

with the defining relations:

$$H_{i,l}v_0 = 0, \qquad \text{for } l > 0, \tag{3.1.2}$$

$$q^{\frac{1}{2}c}v_0 = q^{\frac{1}{2}}v_0. \tag{3.1.3}$$

Note that $\mathcal{F}_n$ is a free S_n^--module of rank 1.

Let $\mathbb{F}$ be a field of characteristic zero and let $\mathfrak{a}$ be an associative $\mathbb{F}$-algebra generated by x_p, y_p $(p \in \mathbb{Z}_{>0})$, z and its inverse z^{-1} with the following relations:

$$[x_p, z] = [y_p, z] = 0,$$

$$[x_p, x_r] = [y_p, y_r] = 0,$$
$$[x_p, y_r] = \delta_{pr} z.$$

Let $\mathfrak{a}^+$ (*resp.* $\mathfrak{a}^-$) be the subalgebra of $\mathfrak{a}$ generated by x_p (*resp.* y_p). We set $\mathfrak{b} = \mathfrak{a}^+ \otimes \mathbb{F}[z, z^{-1}]$. This is a maximal abelian subalgebra of $\mathfrak{a}$. Fix a nonzero scalar $\lambda \in \mathbb{F}^\times$. Let $\mathbb{F}_\lambda$ be the one-dimensional space $\mathbb{F}$ viewed as a $\mathfrak{b}$-module by:

$$z \cdot 1 = \lambda, \quad \mathfrak{a}^+ \cdot 1 = 0.$$

Let $F(\lambda)$ be the induced $\mathfrak{a}$-module

$$F(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}} \mathbb{F}_\lambda = \mathfrak{a} \otimes_{\mathfrak{b}} \mathbb{F}_\lambda.$$

By the defining relations of $\mathfrak{a}$ we obtain an $\mathbb{F}$-linear isomorphism

$$F(\lambda) \cong \mathfrak{a}^-.$$

Since $\mathfrak{a}^-$ is abelian we may regard it as the algebra of polynomials in the variables $y_1, y_2, \cdots$. Then we see that z acts on $\mathfrak{a}^- \cong \mathbb{F}[y_1, y_2, \cdots]$ by the multiplication of λ, x_p acts by $\lambda \frac{\partial}{\partial y_p}$. By this realization we immediately have the following lemma.

Lemma 3.1.1. *$F(\lambda)$ is an irreducible $\mathfrak{a}$-module.*

Fix an skew-symmetric $(n+1) \times (n+1)$-matrix with integral coefficients $M = (m_{ij})_{0 \le i,j \le n}$. We say that $\kappa \in \mathbb{Q}(q)$ is generic with respect to M if for any $k \in \mathbb{Z}_{>0}$ the matrix $([k\langle h_i, \alpha_j\rangle]\kappa^{-km_{ij}})$ is invertible.

Note that if $n = 1$ any κ is not generic with respect to any M. Since the matrix $([k\langle h_i, \alpha_j\rangle])_{0 \le i,j \le n}$ is invertible for $n > 1$, there exists a generic κ for $n > 1$.

Lemma 3.1.2. (1) *For a fixed M, we assume that $\kappa \in \mathbb{Q}(q)$ is generic with respect to M (in particular $n > 1$). Then $\mathcal{F}_n$ is an irreducible S_n-module.*
(2) *$\mathcal{F}_1$ is not irreducible.*

Proof. (1) Set $G(k) = (g(k)_{ij}) = ([k\langle h_i, \alpha_j\rangle]\kappa^{-km_{ij}})$. Since κ is generic with respect to M there exists its inverse $G(k)^{-1} = (g(k)^{ij})$ for any k. Note that by the definition $\sum_{0 \le s \le n} g(k)^{is} g(k)_{sj} = \delta_{ij}$.

We set

$$\tilde{H}_{i,k} = \begin{cases} \sum_{0 \le s \le n} \frac{k}{[k]} g(k)^{is} H_{s,k}, & \text{for } k > 0, \\ H_{i,k}, & \text{for } k < 0. \end{cases}$$

Then we have

$$[\tilde{H}_{i,k}, \tilde{H}_{j,l}] = [\tilde{H}_{i,-k}, \tilde{H}_{j,-l}] = 0,$$
$$[\tilde{H}_{i,k}, \tilde{H}_{j,-l}] = \delta_{k,l}\delta_{ij},$$

for $k, l > 0$. Since all $G(k)$ are regular, $\tilde{H}_{i,k}$ $(0 \le i \le n,\ k > 0)$ generate S_n^+.

We shall use Lemma 3.1.1. Put $\mathbb{F} = \mathbb{Q}(q)$, $\mathfrak{a} = \mathfrak{S}_\mathfrak{n}$, $\mathfrak{a}^\pm = S_n^\pm$, $\lambda = 1$, $x_p = \tilde{H}_{i,k}$ $(k > 0)$, $y_r = \tilde{H}_{j,l}$ $(l < 0)$ where $p = (k-1)(n+1) + i + 1$ and $r = (-l-1)(n+1) + j + 1$.

Then it is clear that $F(1) = \mathcal{F}_n$. By Lemma 3.1.1 we conclude that $\mathcal{F}_n$ is an irreducible S_n-module.

(2) It is easy to see that $\kappa^{-km_{01}} H_{0,-k} + H_{1,-k}$ is a central element of S_1 for each $k \in \mathbb{Z}_{>0}$. Therefore $\mathcal{F}_1$ has infinitely many singular vectors. $\square$

3.2. **Construction of level (1,0) modules.** We assume $n > 1$ and M is an $(n+1) \times (n+1)$-matrix defined as follows;

$$M = \begin{pmatrix} 0 & -a & 0 & \dots & 0 & a \\ a & 0 & -a & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a \\ -a & 0 & 0 & \dots & a & 0 \end{pmatrix}$$

here $a \in \mathbb{Z}$.

Note that one can rewrite $\overline{P} = \oplus_{i=2}^n \mathbb{Z}\overline{\alpha_i} \oplus \mathbb{Z}\overline{\Lambda}_n$. We introduce a twisted version of the group algebra $\mathbb{Q}(q)[\overline{P}]$ by $\mathbb{Z}/2\mathbb{Z}$. We denote it by $\mathbb{Q}(q)\{\overline{P}\}$. This is the $\mathbb{Q}(q)$-algebra generated by symbols $e^{\overline{\alpha_2}}$, $e^{\overline{\alpha_3}}, \cdots, e^{\overline{\alpha_n}}$, $e^{\overline{\Lambda_n}}$ which satisfy the following relations:

$$e^{\overline{\alpha_i}} e^{\overline{\alpha_j}} = (-1)^{\langle \overline{h_i}, \overline{\alpha_j} \rangle} e^{\overline{\alpha_j}} e^{\overline{\alpha_i}} \tag{3.2.1}$$

$$e^{\overline{\alpha_i}} e^{\overline{\Lambda_n}} = (-1)^{\delta_{in}} e^{\overline{\Lambda_n}} e^{\overline{\alpha_i}}. \tag{3.2.2}$$

For $\overline{\alpha} = \sum_{i=2}^n m_i \overline{\alpha_i} + m_{n+1} \overline{\Lambda_n}$ we denote $e^{\overline{\alpha}} = (e^{\overline{\alpha_2}})^{m_2} (e^{\overline{\alpha_3}})^{m_3} \cdots (e^{\overline{\alpha_n}})^{m_n} (e^{\overline{\Lambda_n}})^{m_{n+1}}$. For example $e^{\overline{\alpha_1}} = e^{-2\overline{\alpha_2}} e^{-3\overline{\alpha_3}} \cdots e^{-n\overline{\alpha_n}} e^{(n+1)\overline{\Lambda_n}}$, $e^{\overline{\Lambda_i}} = e^{-\overline{\alpha_{i+1}}} e^{-2\overline{\alpha_{i+2}}} \cdots e^{-(n-i)\overline{\alpha_n}} e^{(n-i+1)\overline{\Lambda_n}}$ where $\overline{\Lambda_i}$ is the i-th fundamental weight. We denote $\overline{\alpha}_0 = -\sum_{i=1}^n \overline{\alpha}_i$ and $\overline{h}_0 = -\sum_{i=1}^n \overline{h}_i$.

Note that $\langle h_i, \alpha_j \rangle = \langle \overline{h}_i, \overline{\alpha}_j \rangle$ for $0 \le i, j \le n$.

We denote by $\mathbb{Q}(q)\{\overline{Q}\}$ the subalgebra of $\mathbb{Q}(q)\{\overline{P}\}$ generated by $e^{\overline{\alpha_i}}$ $(1 \le i \le n)$.

Set
$$W(p)_n = \mathcal{F}_n \otimes \mathbb{Q}(q)\{\overline{Q}\}e^{\overline{\Lambda_p}} \qquad \text{for } 0 \le p \le n.$$
We define the operators $H_{i,l}$, $e^{\overline{\alpha}}$ ($\overline{\alpha} \in \overline{Q}$), $\partial_{\overline{\alpha}_i}$, $z^{H_{i,0}}$, d on $W(p)_n$ for $i = 0, 1, \cdots, n$ as follows:
for $v \otimes e^{\overline{\beta}} = H_{i_1,-k_1} \cdots H_{i_N,-k_N} v_0 \otimes e^{\sum_{j=1}^n m_j \overline{\alpha}_j + \overline{\Lambda}_p} \in W(p)_n$,
$$H_{i,l}(v \otimes e^{\overline{\beta}}) = (H_{i,l}v) \otimes e^{\overline{\beta}},$$
$$e^{\overline{\alpha}}(v \otimes e^{\overline{\beta}}) = v \otimes e^{\overline{\alpha}} e^{\overline{\beta}},$$
$$\partial_{\overline{\alpha_i}}(v \otimes e^{\overline{\beta}}) = \langle \overline{h_i}, \overline{\beta} \rangle v \otimes e^{\overline{\beta}},$$
$$z^{H_{i,0}}(v \otimes e^{\overline{\beta}}) = z^{\langle \overline{h_i}, \overline{\beta} \rangle} \kappa^{\frac{1}{2}\sum_{j=1}^n \langle \overline{h_i}, m_j \overline{\alpha}_j \rangle m_{ij}} v \otimes e^{\overline{\beta}},$$
$$d(v \otimes e^{\overline{\beta}}) = (-\sum_{s=1}^N k_s - \frac{(\overline{\beta}|\overline{\beta})}{2} + \frac{(\overline{\Lambda_p}|\overline{\Lambda_p})}{2}) v \otimes e^{\overline{\beta}}.$$
We have the following lemma.

Lemma 3.2.1. *As operators on $W(p)_n$,*
$$e^{\overline{\alpha}_i} e^{\overline{\alpha}_j} = (-1)^{\langle \overline{h}_i, \overline{\alpha}_j \rangle} e^{\overline{\alpha}_j} e^{\overline{\alpha}_i},$$
$$e^{\overline{\alpha}_i} q^{\partial_{\overline{\alpha}_j}} = q^{\langle \overline{h}_i, \overline{\alpha}_j \rangle} q^{\partial_{\overline{\alpha}_j}} e^{\overline{\alpha}_i},$$
$$e^{\overline{\alpha}_i} z^{\partial_{\overline{\alpha}_j}} = z^{\langle \overline{h}_i, \overline{\alpha}_j \rangle} z^{\partial_{\overline{\alpha}_j}} e^{\overline{\alpha}_i}$$
for $0 \le i, j \le n$.

We introduce the following generating functions:
$$E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k},$$
$$F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k}.$$

Proposition 3.2.2. *Assume $c = 1$ and $n > 1$. Then for each p and κ, the following action gives a $U_q^{\phi'}(\mathfrak{g}_{tor})$-module structure on $W(p)_n$:*
$$q^{\frac{1}{2}c} \mapsto q^{\frac{1}{2}},$$
$$q^{d_1} \mapsto q^d,$$
$$E_i(z) \mapsto \exp(\sum_{k \ge 1} \frac{H_{i,-k}}{[k]} (q^{-1/2} z)^k) \exp(\sum_{k \ge 1} -\frac{H_{i,k}}{[k]} (q^{1/2} z)^{-k}) e^{\overline{\alpha_i}} z^{H_{i,0}+1},$$
$$F_i(z) \mapsto \exp(\sum_{k \ge 1} -\frac{H_{i,-k}}{[k]} (q^{1/2} z)^k) \exp(\sum_{k \ge 1} \frac{H_{i,k}}{[k]} (q^{-1/2} z)^{-k}) e^{-\overline{\alpha_i}} z^{-H_{i,0}+1},$$
$$K_i^+(z) \mapsto \exp((q - q^{-1}) \sum_{k \ge 1} H_{i,k} z^{-k}) q^{\partial_{\overline{\alpha_i}}},$$
$$K_i^-(z) \mapsto \exp(-(q - q^{-1}) \sum_{k \ge 1} H_{i,-k} z^k) q^{-\partial_{\overline{\alpha_i}}}$$

for $0 \leq i \leq n$.

The proof will be given in Appendix.

We have immediately the following lemma.

Lemma 3.2.3. *The* $U_q^{\phi'}(\mathfrak{g}_{tor})$*-module* $W(p)_n$ *is cyclic:* $W(p)_n = U_q^{\phi'}(\mathfrak{g}_{tor})(v_0 \otimes e^{\overline{\Lambda}_p})$.

Theorem 3.2.4. *If* $n > 1$ *and* κ *is generic then* $W(p)_n$ *is irreducible for any* p.

Proof. Since $\mathcal{F}_n$ is irreducible with respect to the action of S_n, it is enough to show that for any non-zero $v = v_0 \otimes \sum_{\overline{\alpha} \in \overline{Q}} a_{\overline{\alpha}} e^{\overline{\alpha}} e^{\overline{\Lambda}_i}$ ($a_{\overline{\alpha}} \in \mathbb{Q}(q)$) there exists $X \in U_q^{\phi'}(\mathfrak{g}_{tor})$ such that $Xv = v_0 \otimes e^{\overline{\Lambda}_p}$. Let $\overline{S_n}$ be the subalgebra of S_n generated by $H_{i,l}$ ($1 \leq i \leq n, l \in \mathbb{Z} \backslash \{0\}$) and $q^{\frac{1}{2}c}$. Let $\overline{\mathcal{F}}_n$ be the $\overline{S_n}$-submodule of $\mathcal{F}_n$ generated by $v_0 \otimes e^{\overline{\Lambda}_i}$, and let $\overline{W(p)_n} = \overline{\mathcal{F}}_n \otimes \mathbb{Q}(q)\{\overline{Q}\} e^{\overline{\Lambda}_p}$. As already known $\overline{W(p)_n}$ is an irreducible $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module. It is obvious that $v \in \overline{W(p)_n}$. Therefore there exists $X \in U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1}) \subset U_q^{\phi'}(\mathfrak{g}_{tor})$ such that $Xv = v_0 \otimes e^{\overline{\Lambda}_p}$. ◻

Remark 3.2.5. Since $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ and $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$ are subalgebras of $U_q^{\phi'}(\mathfrak{g}_{tor})$ we can regard $W(p)_n$ as a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module or as a $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$-module. As a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module, $W(p)_n$ is a level 1 module. On the other hand it is a level 0 module as a $U_q^{(2)'}(\widehat{\mathfrak{sl}}_{n+1})$-module.

3.3. **On the structure of level (1,0) modules.** In this subsection we will study the $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module structure of $W(p)_n$.

Let M be a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module. We denote the character of M by ch_M.

Let $L(\Lambda_p)$ be the irreducible highest weight $U_q(\widehat{\mathfrak{sl}}_{n+1})$-module with highest weight Λ_p. Note that the following identity holds:

$$\mathrm{ch}_{L(\Lambda_p)} = \frac{e^{\Lambda_p} \sum_{\alpha \in \overline{Q}} e^{\alpha - (\frac{1}{2}(\alpha|\alpha) - (\alpha|\overline{\Lambda_p}))\delta}}{\varphi(e^{-\delta})^n}. \tag{3.3.1}$$

Here $\varphi(x) = \prod_{k>0}(1 - x^k)$.

We denote δ_1 by the null root of $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$.

By the definition of $W(p)_n$ and (3.3.1) it is immediate to see the following proposition.

Proposition 3.3.1.

$$\begin{aligned} ch_{W(p)_n} &= \frac{e^{\Lambda_p}\sum_{\alpha\in\overline{Q}} e^{\alpha-(\frac{1}{2}(\alpha|\alpha)-(\alpha|\overline{\Lambda_p}))\delta_1}}{\varphi(e^{-\delta})^n} \\ &= \frac{ch_{L(\Lambda_p)}}{\varphi(e^{-\delta_1})}. \end{aligned}$$

Lemma 3.3.2. *For each $l \in \mathbb{Z}\backslash\{0\}$ there exist $\widehat{H}_l = \sum_{i=0}^{n} a_{i,l}H_{i,l}$ $(a_{i,l} \in \mathbb{Q}(q))$ such that*

$$[\widehat{H}_l, H_{j,k}] = 0$$

for any $1 \leq j \leq n$ and $k \in \mathbb{Z}\backslash\{0\}$. Moreover such $\widehat{H}_l$ is unique up to scalar.

Proof. Note that $\kappa = 1$. The rank of $n\times(n+1)$-matrix $([l\langle h_i, \alpha_j\rangle])_{1\leq i\leq n, 0\leq j\leq n}$ is equal to n. The lemma follows form this fact immediately. $\square$

By the definition of $\widehat{H}_l$ we have

$$[\widehat{H}_k, \widehat{H}_l] = \delta_{k+l,0}\gamma_k, \tag{3.3.2}$$

where $\gamma_k \in \mathbb{Q}(q)$. We fix a normalization of $\widehat{H}_l$ by putting $\gamma_k = 1$ for all k.

Let $\widehat{S}_n$ be the subalgebra of S_n generated by $\widehat{H}_l$. By the definition, $\widehat{S}_n$ acts on $W(p)_n$.

The following two lemmas are easy to see.

Lemma 3.3.3. *For $l > 0$, $\widehat{H}_l(v_0 \otimes e^{\overline{\beta}}) = 0$.*

Lemma 3.3.4. *The action of $U_q^{(1)\prime}(\widehat{\mathfrak{sl}}_{n+1})$ on $W(p)_n$ commutes with the action of $\widehat{S}_n$.*

Let $\widehat{S}_n^-$ be the subalgebra of $\widehat{S}_n$ generated by $\widehat{H}_l$ $(l < 0)$.

Proposition 3.3.5. *As $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module*

$$W(p)_n \cong L(\Lambda_p)^{\oplus\infty}.$$

Proof. Set $\deg(\widehat{H}_k) = k$. Let $M_k = M_k(\widehat{H}_{-1}, \widehat{H}_{-2}, \cdots)$ be a monomial of degree k in variables $\widehat{H}_{-1}, \widehat{H}_{-2}, \cdots$. Then by the above two lemmas, $M_k v_0 \otimes e^{\overline{\Lambda_p}}$ is a singular vector of $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module $W(p)_n$. Let W_{M_k} be the $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-submodule which is generated by $M_k v_0 \otimes e^{\overline{\Lambda_p}}$. Then by the definition of the action of $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$ on $W(p)_n$, we have

$$W_{M_k} \cong L(\Lambda_p - k\delta) \cong L(\Lambda_p).$$

The vectors $\{M_k v_0 \otimes e^{\overline{\Lambda_p}}\}$ are linearly independent. The number of the monomials of degree k is equal to the k-th partition number $p(k)$. Therefore there is a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-submodule W of $W(p)_n$ which is isomorphic to $\oplus_{k\geq 0} L(\Lambda_p - k\delta)^{\oplus p(k)}$. By Proposition 3.3.1 it coincides with $W(p)_n$. This completes proof. $\square$

By Lemma 3.3.4 and the proof of Proposition 3.3.5, the following corollary follows immediately.

Corollary 3.3.6. *As $U_q^{(1)'}(\widehat{\mathfrak{sl}}_{n+1}) \otimes \widehat{S}_n$-module $W(p)_n$ is isomorphic to $L(\Lambda_p) \otimes \widehat{S}_n^-$.*

3.4. **Construction of level (1,1) modules.** We introduce a twisted version of the group algebra $\mathbb{Q}(q)[Q]$ by $\mathbb{Z}\backslash 2\mathbb{Z}$. We denoted it by $\mathbb{Q}(q)\{Q\}$. This is the $\mathbb{Q}(q)$-algebra generated by symbols e^{α_0},$e^{\alpha_1}, \cdots, e^{\alpha_n}$ which satisfy the following relations:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i}. \tag{3.4.1}$$

Similarly to §3.2, we denote $e^{\alpha} = (e^{\alpha_0})^{m_0} (e^{\alpha_1})^{m_1} \cdots (e^{\alpha_n})^{m_n}$ for $\alpha = \sum_{i=0}^n m_i \alpha_i \in Q$.

Let

$$V(p)_n = \mathcal{F}_n \otimes \mathbb{Q}(q)\{Q\} e^{\Lambda_p}.$$

Here we regard e^{Λ_p} only a symbol indexed by p.

We define the operators $H_{i,l}$ $(0 \leq i \leq n,\ l \neq 0)$, e^{α} $(\alpha \in Q)$, ∂_{α_i} and $z^{H_{i,0}}$ $(0 \leq i \leq n)$ on $V(p)_n$ as follows:
for $v \otimes e^{\beta} e^{\Lambda_p} = H_{i_1,-k_1} \cdots H_{i_N,-k_N} v_0 \otimes e^{\beta} e^{\Lambda_p} \in V(p)_n$ $(\beta = \sum_{k=0}^n m_k \alpha_k \in Q)$,

$$\begin{gathered}
H_{i,l}(v \otimes e^{\beta} e^{\Lambda_p}) = (H_{i,l} v) \otimes e^{\beta} e^{\Lambda_p}, \\
e^{\alpha}(v \otimes e^{\beta} e^{\Lambda_p}) = v \otimes (e^{\alpha} e^{\beta}) e^{\Lambda_p}, \\
\partial_{\alpha_i}(v \otimes e^{\beta} e^{\Lambda_p}) = \langle h_i, \beta + \Lambda_p \rangle v \otimes e^{\beta} e^{\Lambda_p}, \\
z^{H_{i,0}}(v \otimes e^{\beta} e^{\Lambda_p}) = z^{\langle h_i, \beta + \Lambda_p \rangle} \kappa^{\frac{1}{2} \sum_{k=0}^n \langle h_i, m_k \alpha_k \rangle m_{ik}} v \otimes e^{\beta} e^{\Lambda_p}. \\
d_1(v \otimes e^{\beta} e^{\Lambda_p}) = (-\sum_{s=1}^N k_s - \frac{(\beta|\beta)}{2} - (\beta|\Lambda_p)) v \otimes e^{\beta} e^{\Lambda_p}, \\
d_2(v \otimes e^{\beta} e^{\Lambda_p}) = m_0 (v \otimes e^{\beta} e^{\Lambda_p}).
\end{gathered}$$

The following lemma is easy.

Lemma 3.4.1. *As operators on $V(p)_n$,*

$$\begin{gathered}
e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i} \\
q^{\partial_{\alpha_i}} e^{\alpha_j} = q^{\langle h_i, \alpha_j \rangle} e^{\alpha_j} q^{\partial_{\alpha_i}} \\
z^{H_{i,0}} e^{\alpha_j} = z^{\langle h_i, \alpha_j \rangle} \kappa^{\frac{1}{2} \langle h_i, \alpha_j \rangle m_{ij}} e^{\alpha_j} z^{H_{i,0}}.
\end{gathered}$$

Proposition 3.4.2. *Assume $c = 1$ and $n > 1$. Then for each p, the following action gives a $U_q(\mathfrak{g}_{tor})$-module structure on $V(p)_n$:*

$$
\begin{aligned}
q^{\frac{1}{2}c} &\mapsto q^{\frac{1}{2}},\\
q^{d_1} &\mapsto q^{d_1}\\
q^{d_2} &\mapsto q^{d_2}\\
E_i(z) &\mapsto \exp\Big(\sum_{k\geq 1}\frac{H_{i,-k}}{[k]}(q^{-1/2}z)^k\Big)\exp\Big(\sum_{k\geq 1}-\frac{H_{i,k}}{[k]}(q^{1/2}z)^{-k}\Big)e^{\alpha_i}z^{H_{i,0}+1},\\
F_i(z) &\mapsto \exp\Big(\sum_{k\geq 1}-\frac{H_{i,-k}}{[k]}(q^{1/2}z)^k\Big)\exp\Big(\sum_{k\geq 1}\frac{H_{i,k}}{[k]}(q^{-1/2}z)^{-k}\Big)e^{-\alpha_i}z^{-H_{i,0}+1},\\
K_i^+(z) &\mapsto \exp\Big((q-q^{-1})\sum_{k\geq 1}H_{i,k}z^{-k}\Big)q^{\partial_{\alpha_i}},\\
K_i^-(z) &\mapsto \exp\Big(-(q-q^{-1})\sum_{k\geq 1}H_{i,-k}z^{k}\Big)q^{-\partial_{\alpha_i}}
\end{aligned}
$$

for $0 \leq i \leq n$.

The proof will be given in Appendix.

It is easy to see the following lemma.

Lemma 3.4.3. *$V(p)_n$ is a cyclic $U_q(\mathfrak{g}_{tor})$-module: $V(p)_n = U_q(\mathfrak{g}_{tor})(v_0\otimes e^{\Lambda_p})$.*

Lemma 3.4.4. *$V(p)_n$ has level 1 as a $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module and as a $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$-module.*

Proof. It is clear that $V(p)_n$ is a level 1 $U_q^{(1)}(\widehat{\mathfrak{sl}}_{n+1})$-module. The center of $U_q^{(2)}(\widehat{\mathfrak{sl}}_{n+1})$ is $\prod_{k=0}^n K_i$. By the definition it acts as the scalar q on $V(p)_n$. $\square$

4. On $U_q(\mathfrak{sl}_{2,tor})$

4.1. In this section we assume that $\mathfrak{g} = \mathfrak{sl}_2$. We shall try to find finitely many generators of $U_q(\mathfrak{sl}_{2,tor})$.

Let

$$
\begin{gathered}
E_i = E_{i,0},\quad F_i = F_{i,0},\quad q^{\pm h_i} = K_i^{\pm},\ \text{for } i = 0,1,\\
E_{-1} = F_{0,1}K_0^-,\quad F_{-1} = K_0^+E_{0,-1},\quad q^{\pm h_{-1}} = q^{\pm c}K_0^{\mp}.
\end{gathered}
$$

Proposition 4.1.1. *$U_q(\mathfrak{sl}_{2,tor})$ is generated by E_i, F_i, $q^{\pm h_i}$ $(i = -1, 0, 1)$, $q^{\pm\frac{1}{2}c}$, $q^{\pm d_1}$, $q^{\pm d_2}$.*

Proof. Let $\mathcal{A}$ be the subalgebra of $U_q(\mathfrak{sl}_{2,tor})$ generated by E_i, F_i, $q^{\pm h_i}$ ($i=-1,0,1$), $q^{\pm\frac{1}{2}c}$, $q^{\pm d_1}$, $q^{\pm d_2}$. By the definition we have $E_{0,-1} = q^{-h_0}F_{-1}$ and $F_{0,1} = E_{-1}q^{h_0}$. Since

$$\begin{aligned}[E_{0,0}, F_{0,1}] &= \frac{1}{q-q^{-1}} q^{-\frac{1}{2}c} K^+_{0,1} \\ &= q^{-\frac{1}{2}c} q^{h_0} H_{0,1}\end{aligned}$$

and

$$\begin{aligned}[E_{0,-1}, F_{0,0}] &= -\frac{1}{q-q^{-1}} q^{\frac{1}{2}c} K^-_{0,-1} \\ &= q^{\frac{1}{2}c} q^{-h_0} H_{0,-1}.\end{aligned}$$

we deduce $H_{0,1}$ and $H_{0,-1} \in \mathcal{A}$. We recall (2.2.7)

$$[H_{i,k}, E_{j,l}] = \frac{1}{k}[k\langle h_i, \alpha_j\rangle] q^{-\frac{1}{2}|k|c} \kappa^{-km_{ij}} E_{j,k+l},$$

$$[H_{i,k}, F_{j,l}] = -\frac{1}{k}[k\langle h_i, \alpha_j\rangle] q^{\frac{1}{2}|k|c} \kappa^{-km_{ij}} F_{j,k+l}.$$

By these formulas we have $E_{i,k}$, $F_{i,k} \in \mathcal{A}$ for $i=0,1$, $k \in \mathbb{Z}$ inductively.

On the other hand we know

$$\begin{aligned}[E_{0,1}, F_{0,1}] &= \frac{1}{q-q^{-1}} q^{-\frac{1}{2}c} K^+_{0,2} \\ &= \frac{1}{2}(q-q^{-1}) H^2_{0,1} + H_{0,2}.\end{aligned}$$

Therefore we get $H_{0,2} \in \mathcal{A}$. Similarly we have $H_{i,l} \in \mathcal{A}$ for any i, l.

This completes the proof. □

Lemma 4.1.2. *The following relations hold in $U_q(\mathfrak{sl}_{2,tor})$:*

$$[q^{\pm h_i}, q^{\pm h_j}] = 0, \tag{4.1.1}$$

$$[q^{\pm d_i}, q^{\pm h_j}] = 0, \tag{4.1.2}$$

$$\begin{aligned} q^{d_1} E_j q^{-d_1} &= q^{\delta_{j,-1}} E_j, \\ q^{d_1} F_j q^{-d_1} &= q^{-\delta_{j,-1}} F_j, \end{aligned} \tag{4.1.3}$$

$$\begin{aligned} q^{d_2} E_j q^{-d_2} &= q^{-1+\delta_{j,1}} E_j, \\ q^{d_2} F_j q^{-d_2} &= q^{1-\delta_{j,1}} F_j, \end{aligned} \tag{4.1.4}$$

$$\begin{aligned} q^{h_i} E_j q^{-h_i} &= q^{a_{ij}} E_j, \\ q^{h_i} F_j q^{-h_i} &= q^{-a_{ij}} F_j, \end{aligned} \tag{4.1.5}$$

where

$$(a_{ij})_{-1\le i,j\le 1} = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix},$$

$$[E_i, F_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \qquad \textit{for } |i-j| \le 1, \tag{4.1.6}$$

$$E_{-1}^3F_1 - q^{-2}[3]E_{-1}^2F_1E_{-1} + q^{-4}[3]E_{-1}F_1E_{-1}^2 - q^{-6}F_1E_{-1}^3 = 0, \tag{4.1.7}$$

$$E_1^3F_{-1} - q^2[3]E_1^2F_{-1}E_1 + q^4[3]E_1F_{-1}E_1^2 - q^6F_{-1}E_1^3 = 0, \tag{4.1.8}$$

$$F_{-1}^3E_1 - q^{-2}[3]F_{-1}^2E_1F_{-1} + q^{-4}[3]F_{-1}E_1F_{-1}^2 - q^{-6}E_1F_{-1}^3 = 0, \tag{4.1.9}$$

$$F_1^3E_{-1} - q^2[3]F_1^2E_{-1}F_1 + q^4[3]F_1E_{-1}F_1^2 - q^6E_{-1}F_1^3 = 0, \tag{4.1.10}$$

$$E_i^3E_j - [3]E_i^2E_jE_i + [3]E_iE_jE_i^2 - E_jE_i^3 = 0, \qquad \textit{for } |i-j| = 1, \tag{4.1.11}$$

$$F_i^3F_j - [3]F_i^2F_jF_i + [3]F_iF_jF_i^2 - F_jF_i^3 = 0, \qquad \textit{for } |i-j| = 1, \tag{4.1.12}$$

$$E_{-1}E_1 - q^2E_1E_{-1} = 0, \tag{4.1.13}$$

$$F_{-1}F_1 - q^2F_1F_{-1} = 0. \tag{4.1.14}$$

Proof. By the definition of $U_q(\mathfrak{g}_{tor})$ and E_i, F_i and q^{h_i} it is easy to check these relations. ☐

Let $\mathcal{U}$ be an associative algebra over $\mathbb{Q}(q)$ generated by E_i, F_i, $q^{\pm h_i}$ $(i = -1, 0, 1)$, $q^{\pm\frac{1}{2}c}$, $q^{\pm d_1}$, $q^{\pm d_2}$ with relations (4.1.1)-(4.1.14). Then we have,

Corollary 4.1.3. *There is a canonical surjective algebra homomorphism* $\Psi : \mathcal{U} \to U_q(\mathfrak{sl}_{2,tor})$.

Remark 4.1.4. Ψ has a highly nontrivial kernel. It is important to decide it. For example the following formulas holds in $U_q(\mathfrak{sl}_{2,tor})$:

$$\kappa^{m_{01}} E_{0,0}E_{1,-1} - q^{-2}\kappa^{m_{01}} E_{1,-1}E_{0,0} = q^{-2}E_{0,-1}E_{1,0} - E_{1,0}E_{0,-1},$$

$$E_{0,-1} = q^{-h_0}F_{-1}$$

and

$$E_{1,-1} = \frac{\kappa^{-m_{01}}}{[-2]}[F_{-1}F_0 - q^{-2}F_0F_{-1}, E_1].$$

Therefore we have

$$\begin{aligned} X = {} & \frac{1}{[-2]}E_0[F_{-1}F_0 - q^{-2}F_0F_{-1}, E_1] - \frac{q^{-2}}{[-2]}[F_{-1}F_0 - q^{-2}F_0F_{-1}, E_1]E_0 \\ & - q^{-2}q^{-h_0}F_{-1}E_1 - E_1q^{-h_0}F_{-1} \\ = {} & 0 \end{aligned}$$

in $U_q(\mathfrak{sl}_{2,tor})$. Thus $X \in \mathrm{Ker}\Psi$. But, as an element of $\mathcal{U}$, X is not equal to 0.

4.2. Let

$$E_{0^*} = F_{1,1}K_1^-, \qquad F_{0^*} = K_1^+E_{1,-1}, \qquad q^{\pm h_{0^*}} = q^{\pm c}K_1^{\mp}.$$

Proposition 4.2.1. *The subalgebra generated by E_i, F_i and q^{h_i} for $i = 0, 1$, E_{0^*}, F_{0^*}, $q^{\pm h_{0^*}}$, $q^{\pm c}$, $q^{\pm d_1}$, $q^{\pm d_2}$ is equal to $U_q(\mathfrak{sl}_{2,tor})$. That is, they are generators of $U_q(\mathfrak{sl}_{2,tor})$. Moreover these generators satisfy relations similar to the ones in Lemma 4.1.2.*

Proof. This proposition is proved in the same way as Proposition 4.1.1 and Lemma 4.1.2. □

We have immediately the following lemma.

Lemma 4.2.2. *Let $U_q^{(1)}(\widehat{\mathfrak{sl}}_2)$ be the subalgebras generated by E_i, F_i, q^{h_i} for $i = 1, 0^*$ and $q^{\pm(d_1+d_2)}$, $U_q^{(2)}(\widehat{\mathfrak{sl}}_2)$ the subalgebras generated by E_i, F_i, q^{h_i} for $i = 0, 1$ and $q^{\pm(d_1+d_2)}$, $U_q^{(3)}(\widehat{\mathfrak{sl}}_2)$ the subalgebras generated by E_i, F_i, $q^{\pm h_i}$ for $i = 0^*, -1$ and $q^{\pm(d_1+d_2)}$, and $U_q^{(4)}(\widehat{\mathfrak{sl}}_2)$ the subalgebras generated by E_i, F_i, $q^{\pm h_i}$ for $i = 0, -1$ and $q^{\pm(d_1+d_2)}$. Then $U_q^{(i)}$ $(i = 1, 2, 3, 4)$ are isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$.*

Those four algebras are schematically visualized by Fig. 1.

$0 \qquad U_q^{(2)} \qquad 1$

$U_q^{(4)} \qquad\qquad U_q^{(1)}$

$-1 \qquad U_q^{(3)} \qquad 0^*$

Fig. 1.

Let $U_q(\mathfrak{sl}_2)_{(\mathfrak{i})}$ ($i = -1, 0, 1, 0^*$) be the sub algebra of $U_q(\mathfrak{sl}_{2,\mathfrak{tor}})$ generated by E_i, F_i, $q^{\pm h_i}$. All $U_q(\mathfrak{sl}_{2,(\mathfrak{i})})$ ($i \in \{-1, 0, 1, 0^*\}$) are isomorphic to $U_q(\mathfrak{sl}_2)$. The upper left circle in Fig. 1 means $U_q(\mathfrak{sl}_2)_{(\mathfrak{o})}$, the upper right one means $U_q(\mathfrak{sl}_2)_{(\mathfrak{1})}$, the lower left one means $U_q(\mathfrak{sl}_2)_{(-\mathfrak{1})}$ and the lower right one means $U_q(\mathfrak{sl}_2)_{(\mathfrak{o}^*)}$. The diagram

$$\circ^i \Longleftrightarrow \circ^j \qquad (i, j \in \{-1, 0, 1, 0^*\})$$

means the algebra generated by $U_q(\mathfrak{sl}_2)_{(\mathfrak{i})}$ and $U_q(\mathfrak{sl}_2)_{(\mathfrak{j})}$ is isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$. For example $\circ^0 \Longleftrightarrow \circ^1$ means the algebra generated by $U_q(\mathfrak{sl}_2)_{(\mathfrak{o})}$ and $U_q(\mathfrak{sl}_2)_{(\mathfrak{1})}$ which we call $U_q^{(2)}(\widehat{\mathfrak{sl}}_2)$ is isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$. The meaning of the diagram

$$\circ^i \Longrightarrow \circ^j$$

is as follows: In the algebra generated by $U_q(\mathfrak{sl}_2)_{(\mathfrak{i})}$ and $U_q(\mathfrak{sl}_2)_{(\mathfrak{j})}$, the following relations hold

$$q^{h_i} E_j q^{-h_i} = q^2 E_j, \qquad q^{h_j} E_i q^{-h_j} = q^2 E_i, \tag{4.2.1}$$

$$q^{h_i} F_j q^{-h_i} = q^{-2} F_j, \qquad q^{h_j} F_i q^{-h_j} = q^{-2} F_i, \tag{4.2.2}$$

$$\begin{gathered} E_i^3 F_j - q^{-2}[3] E_i^2 F_j E_i + q^{-4}[3] E_i F_j E_i^2 - q^{-6} F_j E_i^3 = 0, \\ E_j^3 F_i - q^2[3] E_j^2 F_i E_j + q^4[3] E_j F_i E_j^2 - q^6 F_i E_j^3 = 0, \end{gathered} \tag{4.2.3}$$

$$\begin{gathered} F_i^3 E_j - q^{-2}[3] F_i^2 E_j F_i + q^{-4}[3] F_i E_j F_i^2 - q^{-6} E_j F_i^3 = 0, \\ F_j^3 E_i - q^2[3] F_j^2 E_i F_j + q^4[3] F_j E_i F_j^2 - q^6 E_i F_j^3 = 0, \end{gathered} \tag{4.2.4}$$

$$E_i E_j - q^2 E_j E_i = 0, \tag{4.2.5}$$

$$F_i F_j - q^2 F_j F_i = 0. \tag{4.2.6}$$

Appendix A.

A.1. Proof of Proposition 3.2.2 and 3.4.2. For the proof we rewrite the defining relation of $U_q(\mathfrak{g}_{tor})$ generating function level.

$$q^{\pm\frac{1}{2}c} \text{ are central,} \tag{A.1.1}$$

$$K_i^+ K_i^- = K_i^- K_i^+ = 1, \tag{A.1.2}$$

$$K_i^{\pm}(z) K_j^{\pm}(w) = K_j^{\pm}(w) K_i^{\pm}(z) \tag{A.1.3}$$

$$\theta_{-\langle h_i,\alpha_j\rangle}(q^{-c}\kappa^{m_{ij}}\frac{z}{w}) K_i^-(z) K_j^+(w) = \theta_{-\langle h_i,\alpha_j\rangle}(q^{c}\kappa^{m_{ij}}\frac{z}{w}) K_j^+(w) K_i^-(z) \tag{A.1.4}$$

$$q^{d_1} K_j^{\pm}(z) q^{-d_1} = K_j^{\pm}(q^{-1}z), \tag{A.1.5}$$

$$[q^{d_2}, K_j^{\pm}(z)] = 0, \tag{A.1.6}$$

$$q^{d_1} E_j(z) q^{-d_1} = E_j(q^{-1}z), \tag{A.1.7}$$
$$q^{d_1} F_j(z) q^{-d_1} = F_j(q^{-1}z),$$

$$q^{d_2} E_j(z) q^{-d_2} = q^{\delta_{j0}} E_j(z), \tag{A.1.8}$$
$$q^{d_2} F_j(z) q^{-d_2} = q^{-\delta_{j0}} F_j(z),$$

$$K_i^+(z) E_j(w) = \theta_{-\langle h_i,\alpha_j\rangle}(q^{-\frac{1}{2}c}\kappa^{-m_{ij}}\frac{w}{z}) E_j(w) K_i^+(z) \tag{A.1.9}$$

$$K_i^-(z) E_j(w) = \theta_{\langle h_i,\alpha_j\rangle}(q^{-\frac{1}{2}c}\kappa^{m_{ij}}\frac{z}{w}) E_j(w) K_i^-(z)$$
$$K_i^+(z) F_j(w) = \theta_{\langle h_i,\alpha_j\rangle}(q^{\frac{1}{2}c}\kappa^{-m_{ij}}\frac{w}{z}) F_j(w) K_i^+(z)$$
$$K_i^-(z) F_j(w) = \theta_{-\langle h_i,\alpha_j\rangle}(q^{\frac{1}{2}c}\kappa^{m_{ij}}\frac{z}{w}) F_j(w) K_i^-(z)$$

$$[E_i(z), F_j(w)] = \delta_{i,j}\frac{1}{q-q^{-1}}\{\delta(q^c\frac{w}{z}) K_i^+(q^{\frac{1}{2}c}w) - \delta(q^c\frac{z}{w}) K_i^-(q^{\frac{1}{2}c}z)\} \tag{A.1.10}$$

$$(\kappa^{m_{ij}}z - q^{\langle h_i,\alpha_j\rangle}w) E_i(z) E_j(w) = (q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z - w) E_j(w) E_i(z) \tag{A.1.11}$$

$$(\kappa^{m_{ij}}z - q^{-\langle h_i,\alpha_j\rangle}w)F_i(z)F_j(w) = (q^{-\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z - w)F_j(w)F_i(z)$$

$$\sum_{\sigma\in\mathfrak{S}_m}\sum_{r=0}^{m}(-1)^r \begin{bmatrix} m \\ r \end{bmatrix}_q E_i(z_{\sigma(1)})\cdots E_i(z_{\sigma(r)})E_j(w)E_i(z_{\sigma(r+1)})\cdots E_i(z_{\sigma(m)}) = 0 \tag{A.1.12}$$

$$\sum_{\sigma\in\mathfrak{S}_m}\sum_{r=0}^{m}(-1)^r \begin{bmatrix} m \\ r \end{bmatrix}_q F_i(z_{\sigma(1)})\cdots F_i(z_{\sigma(r)})F_j(w)F_i(z_{\sigma(r+1)})\cdots F_i(z_{\sigma(m)})$$

where $i \neq j$ and $m = 1 - \langle h_i, \alpha_j\rangle$.

In these formulas we denote $\theta_m(z) = \frac{zq^m-1}{z-q^m}$ for $m \in \mathbb{Z}$, $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$.

If Proposition 3.4.2 holds, then, from Lemma 3.2.1 and 3.4.1, we have Proposition 3.2.2 by putting $h_i \mapsto \overline{h_i}$, $\alpha_i \mapsto \overline{\alpha}_i$, $\kappa \mapsto 1$ and $z^{H_{i,0}} \mapsto z^{\overline{\partial}_i}$. Therefore it is enough to show Proposition 3.4.2.

The relations (A.1.1), (A.1.2) and (A.1.3) are trivial. (A.1.4) is just the commutation relations of Heisenberg algebra S_n. Therefore, by the definition of $V(p)_n$, it is clear that (A.1.4) holds. The relations (A.1.5), (A.1.6) immediately follows from the definition of d_1 and d_2.

Let us show (A.1.7) and (A.1.8). Take $v \otimes e^{\beta}e^{\Lambda_p} \in V(p)_n$ where $\beta = \sum_{k=0}^{n} m_k\alpha_k \in Q$. Then we have

$$\begin{aligned}
&q^{d_1}e^{\pm\alpha_j}z^{\pm H_{j,0}+1}q^{-d_1}(v\otimes e^{\beta}e^{\Lambda_p}) \\
&= q^{\mp(\alpha_j|\beta+\Lambda_p)-1}z^{\pm\langle h_j,\beta+\Lambda_p\rangle+1}\kappa^{\frac{1}{2}\sum_{k=0}^{n}\langle h_j,m_k\alpha_k\rangle m_{jk}}v\otimes e^{\pm\alpha_j}e^{\beta}e^{\Lambda_p} \\
&= (q^{-1}z)^{\pm\langle h_j,\beta+\Lambda_p\rangle+1}\kappa^{\frac{1}{2}\sum_{k=0}^{n}\langle h_j,m_k\alpha_k\rangle m_{jk}}v\otimes e^{\pm\alpha_j}e^{\beta}e^{\Lambda_p} \\
&= e^{\pm\alpha_j}(q^{-1}z)^{\pm H_{j,0}+1}(v\otimes e^{\beta}e^{\Lambda_p}).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
q^{d_1}E_j(z)q^{-d_1} &= q^{d_1}\exp(\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{-\frac{1}{2}k}z^k)\exp(-\sum_{k\geq 1}\frac{H_{j,k}}{[k]}q^{-\frac{1}{2}k}z^{-k})e^{\alpha_j}z^{H_{j,0}+1}q^{-d_1} \\
&= \exp(\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{-\frac{1}{2}k}(q^{-1}z)^k)\exp(-\sum_{k\geq 1}\frac{H_{j,k}}{[k]}q^{-\frac{1}{2}k}(q^{-1}z)^{-k})e^{\alpha_j}(q^{-1}z)^{H_{j,0}+1} \\
&= E_j(q^{-1}z).
\end{aligned}$$

Similarly we have $q^{d_1}F_j(z)q^{-d_1} = F_j(q^{-1}z)$.

It is clear that

$$q^{d_2}e^{\pm\alpha_j}z^{\pm H_{j,0}+1}q^{-d_2}(v\otimes e^{\beta}e^{\Lambda_p}) = q^{\pm\delta_{j0}}e^{\pm\alpha_j}z^{\pm H_{j,0}+1}(v\otimes e^{\beta}e^{\Lambda_p}). \tag{A.1.13}$$

From (A.1.13) and the fact that q^{d_2} commutes with $H_{j,k}$, we have (A.1.8).

We shall show (A.1.9). We denote

$$E_i^+(z) = \exp(\sum_{k\geq 1}\frac{H_{i,-k}}{[k]}q^{-\frac{1}{2}k}z^k),$$

$$E_i^-(z) = \exp(-\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{-\frac{1}{2}k}z^{-k}),$$

$$F_i^+(z) = \exp(-\sum_{k\geq 1}\frac{H_{i,-k}}{[k]}q^{\frac{1}{2}k}z^k),$$

$$F_i^-(z) = \exp(\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{\frac{1}{2}k}w^{-k}).$$

Let us proof

$$K_i^+(z)E_j(w) = \theta_{-\langle h_i,\alpha_j\rangle}(q^{-\frac{1}{2}}\kappa^{m-ij}\frac{w}{z})E_j(w)K_i^+(z)$$

We have

$$\begin{aligned}[(q-q^{-1})\sum_{k\geq 1}H_{i.k}z^{-k},\sum_{l\geq 1}\frac{H_{j.-l}}{[l]}q^{-\frac{1}{2}l}w^l] &= \sum_{k,l}(q-q^{-1})\frac{1}{[l]}[H_{i.k},H_{j.-l}]z^{-k}q^{-\frac{1}{2}l}w^l\\ &= \sum_{k}(q-q^{-1})\frac{[k\langle h_i,\alpha_j\rangle]}{k}\kappa^{-km_{ij}}q^{-\frac{1}{2}k}(\frac{w}{z})^k\\ &= \sum_{k}\frac{1}{k}(q^{k(\langle h_i,\alpha_j\rangle-\frac{1}{2})}-q^{k(-\langle h_i,\alpha_j\rangle-\frac{1}{2})})\kappa^{-km_{ij}}(\frac{w}{z})^k\\ &= \log\frac{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}{1-q^{\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}\end{aligned}$$

and

$$[(q-q^{-1})\sum_{k\geq 1}H_{i.k}z^{-k},-\sum_{l\geq 1}\frac{H_{j.l}}{[l]}q^{-\frac{1}{2}l}w^{-l}] = 0.$$

We recall Campbell-Hausdorff formula: let A and B be noncommutative operators and $C=[A,B]$. If $[C,A]=[C,B]=0$ then we have $e^Ae^B = e^Ce^Be^A$.

By Campbell-Hausdorff formula we get

$$\begin{aligned}&\exp((q-q^{-1})\sum_{k\geq 1}H_{i.k}z^{-k})E_j^+(w)E_j^-(w)\\&=\frac{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}{1-q^{\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}E_j^+(w)E_j^-(w)\exp((q-q^{-1})\sum_{k\geq 1}H_{i.k}z^{-k}).\end{aligned}$$

On the other hand, by Lemma 3.4.1 we have

$$q^{\partial_{\alpha_i}}e^{\alpha_j}w^{H_{j,0}}=q^{\langle h_i,\alpha_j\rangle}e^{\alpha_j}w^{H_{j,0}}q^{\partial_{\alpha_i}}.$$

Thus we get

$$\begin{aligned}&K_i^+(z)E_j(w)\\&=\exp((q-q^{-1})\sum_{k\geq 1}H_{i.k}z^{-k})E_j^+(w)E_j^-(w)q^{\partial_{\alpha_i}}e^{\alpha_j}w^{H_{j,0}+1}\\&=\frac{1-q^{-\langle h_i,\alpha_j\rangle-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}{q^{-\langle h_i,\alpha_j\rangle}-q^{-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z}}E_j(w)K_i^+(z)\\&=\theta_{-\langle h_i,\alpha_j\rangle}(q^{-\frac{1}{2}}\kappa^{-m_{ij}}\frac{w}{z})E_j(w)K_i^+(z).\end{aligned}$$

The other formulas in (A.1.9) can be checked by similar arguments.

Let us show (A.1.10). We have

$$\begin{aligned}&[-\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{-\frac{1}{2}k}z^{-k},-\sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{\frac{1}{2}k}w^{k}]\\&=\sum_{k\geq 1}\frac{1}{k[k]}[k\langle h_i,\alpha_j\rangle]\kappa^{-m_{ij}}(\frac{w}{z})^k\\&=\begin{cases}\log\frac{1}{(1-q\frac{w}{z})(1-q^{-1}\frac{w}{z})}, & \langle h_i,\alpha_j\rangle=2 \quad (i=j),\\ \log(1-\kappa^{-m_{ij}}\frac{w}{z}), & \langle h_i,\alpha_j\rangle=-1,\\ 0, & \langle h_i,\alpha_j\rangle=0.\end{cases}\end{aligned}$$

For example we will show in the case of $\langle h_i,\alpha_j\rangle=-1$. By Campbell-Hausdorff formula we get

$$E_i^{\pm}(z)F_j^{\pm}(w)=F_j^{\pm}(w)E_i^{\pm}(z)$$

and

$$E_i^-(z)F_j^+(w)=(1-\kappa^{-m_{ij}}\frac{w}{z})F_j^+(w)E_i^-(z).$$

On the other hand, by Lemma 3.4.1, we have

$$z^{H_{i,0}}e^{-\alpha_j}=z\kappa^{\frac{1}{2}m_{ij}}e^{-\alpha_j}z^{H_{i,0}}.$$

Therefore we get

$$\begin{aligned}E_i(z)F_j(w) &= E_i^+(z)E_i^-(z)e^{\alpha_i}z^{H_{i,0}+1}F_j^+(w)F_j^-(w)e^{-\alpha_j}w^{-H_{j,0}+1}\\ &= (z\kappa^{\frac{1}{2}m_{ij}} - w\kappa^{-\frac{1}{2}m_{ij}})E_i^+(z)F_j^+(w)E_i^-(z)F_j^-(w)\\ &\quad \times e^{\alpha_i}e^{-\alpha_j}z^{H_{i,0}+1}w^{-H_{j,0}+1}.\end{aligned}$$

By a similar argument we have

$$\begin{aligned}F_j(w)E_i(z) = -(w\kappa^{-\frac{1}{2}m_{ij}} - z\kappa^{\frac{1}{2}m_{ij}})E_i^+(z)F_j^+(w)E_i^-(z)F_j^-(w)\\ \times e^{\alpha_i}e^{-\alpha_j}z^{H_{i,0}+1}w^{-H_{j,0}+1}.\end{aligned}$$

Therefore we get

$$[E_i(z), F_j(w)] = 0.$$

Similarly one can check the other formulas.

We will show (A.1.11). We have

$$\begin{aligned}&[-\sum_{k\geq 1}\frac{H_{i,k}}{[k]}q^{-\frac{1}{2}k}z^{-k}, \sum_{k\geq 1}\frac{H_{j,-k}}{[k]}q^{-\frac{1}{2}k}w^{k}]\\ &= \sum_{k\geq 1}\frac{1}{k[k]}[k\langle h_i,\alpha_j\rangle]\kappa^{-m_{ij}}(\frac{w}{z})^k\\ &= \begin{cases}\log(1-\frac{w}{z})(1-q^{-2}\frac{w}{z}), & i=j,\\ \log\frac{1}{1-q^{-1}\kappa^{-m_{ij}}\frac{w}{z}}, & \langle h_i,\alpha_j\rangle = -1,\\ 0, & \langle h_i,\alpha_j\rangle = 0.\end{cases}\end{aligned}$$

For example let us show in the case of $\langle h_i,\alpha_j\rangle = -1$. If $\langle h_i,\alpha_j\rangle \neq -1$ one can show the formula by a similar argument. By Lemma 3.4.1 we get

$$\begin{aligned}&(z\kappa^{m_{ij}} - q^{\langle h_i,\alpha_j\rangle}w)E_i(z)E_j(w)\\ &= (z\kappa^{m_{ij}} - q^{-1}w)E_i^+(z)E_i^-(z)e^{\alpha_i}z^{H_{i,0}+1}E_j^+(w)E_j^-(w)e^{\alpha_j}w^{H_{j,0}+1}\\ &= \frac{z\kappa^{m_{ij}} - q^{-1}w}{z\kappa^{\frac{1}{2}m_{ij}}(1-q^{-1}\kappa^{-m_{ij}}\frac{w}{z})}E_i^+(z)E_j^+(w)E_i^-(z)E_j^-(w)\\ &\qquad \times e^{\alpha_i}e^{\alpha_j}z^{H_{i,0}+1}w^{H_{j,0}+1}\\ &= \kappa^{\frac{1}{2}m_{ij}}E_i^+(z)E_j^+(w)E_i^-(z)E_j^-(w)e^{\alpha_i}e^{\alpha_j}z^{H_{i,0}+1}w^{H_{j,0}+1}.\end{aligned}$$

On the other hand we have

$$
\begin{aligned}
&(q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z-w)E_j(w)E_i(z)\\
&=\frac{q^{-1}\kappa^{m_{ij}}z-w}{w\kappa^{-\frac{1}{2}m_{ij}}(1-q^{-1}\kappa^{m_{ij}}\frac{z}{w})}E_j^+(w)E_i^+(z)E_j^-(w)E_i^-(z)\\
&\qquad\times e^{\alpha_j}e^{\alpha_i}w^{H_{j,0}+1}z^{H_{i,0}+1}\\
&=\kappa^{\frac{1}{2}m_{ij}}E_i^+(z)E_j^+(w)E_i^-(z)E_j^-(w)e^{\alpha_i}e^{\alpha_j}z^{H_{i,0}+1}w^{H_{j,0}+1}.
\end{aligned}
$$

Thus we have $(z\kappa^{m_{ij}}-q^{\langle h_i,\alpha_j\rangle}w)E_i(z)E_j(w)=(q^{\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z-w)E_j(w)E_i(z)$.

The formula $(z\kappa^{m_{ij}}-q^{-\langle h_i,\alpha_j\rangle}w)F_i(z)F_j(w)=(q^{-\langle h_i,\alpha_j\rangle}\kappa^{m_{ij}}z-w)F_j(w)F_i(z)$ is proved similarly.

Let us prove (A.1.12). Assume that $\langle h_i,\alpha_j\rangle=-1$. This is the most complicated case. The other cases can be proved similarly.

We have the following formulas:

$$
\begin{aligned}
&E_i(z_1)E_i(z_2)E_j(w)\\
&=\frac{(z_1-z_2)(z_1-q^{-2}z_2)\kappa^{-m_{ij}}}{z_1z_2(1-\frac{q^{-1}\kappa^{-m_{ij}}w}{z_1})(1-\frac{q^{-1}\kappa^{-m_{ij}}w}{z_2})}E_i^+(z_1)E_i^+(z_2)E_j^+(w)E_i^-(z_1)E_i^-(z_2)E_j^-(w)\\
&\qquad\times e^{2\alpha_i}e^{\alpha_j}z_1^{H_{i,0}+1}z_2^{H_{i,0}+1}w^{H_{j,0}+1},
\end{aligned}
$$

$$
\begin{aligned}
&E_i(z_1)E_j(w)E_i(z_2)\\
&=\frac{(z_1-z_2)(z_1-q^{-2}z_2)}{z_1w(1-\frac{q^{-1}\kappa^{-m_{ij}}w}{z_1})(1-\frac{q^{-1}\kappa^{m_{ij}}z_2}{w})}E_i^+(z_1)E_i^+(z_2)E_j^+(w)E_i^-(z_1)E_i^-(z_2)E_j^-(w)\\
&\qquad\times e^{2\alpha_i}e^{\alpha_j}z_1^{H_{i,0}+1}z_2^{H_{i,0}+1}w^{H_{j,0}+1},
\end{aligned}
$$

$$
\begin{aligned}
&E_j(w)E_i(z_1)E_i(z_2)\\
&=\frac{(z_1-z_2)(z_1-q^{-2}z_2)\kappa^{m_{ij}}}{w^2(1-\frac{q^{-1}\kappa^{m_{ij}}z_1}{w})(1-\frac{q^{-1}\kappa^{m_{ij}}z_2}{w})}E_i^+(z_1)E_i^+(z_2)E_j^+(w)E_i^-(z_1)E_i^-(z_2)E_j^-(w)\\
&\qquad\times e^{2\alpha_i}e^{\alpha_j}z_1^{H_{i,0}+1}z_2^{H_{i,0}+1}w^{H_{j,0}+1}.
\end{aligned}
$$

Therefore it is enough to show that

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_2}(z_{\sigma(1)}-z_{\sigma(2)})(z_{\sigma(1)}-q^{-2}z_{\sigma(2)})\\
&\quad\times\{\frac{\kappa^{-m_{ij}}}{z_{\sigma(1)}z_{\sigma(2)}(1-\frac{q^{-1}\kappa^{-m_{ij}}w}{z_{\sigma(1)}})(1-\frac{q^{-1}\kappa^{-m_{ij}}w}{z_{\sigma(2)}})}-\frac{q+q^{-1}}{z_{\sigma(1)}w(1-\frac{q^{-1}\kappa^{-m_{ij}}w}{z_{\sigma(1)}})(1-\frac{q^{-1}\kappa^{m_{ij}}z_{\sigma(2)}}{w})}\\
&\quad+\frac{\kappa^{m_{ij}}}{w^2(1-\frac{q^{-1}\kappa^{m_{ij}}z_{\sigma(1)}}{w})(1-\frac{q^{-1}\kappa^{m_{ij}}z_{\sigma(2)}}{w})}\}\\
&=0.
\end{aligned}\tag{A.1.14}$$

This identity is proved by a direct calculation.
Thus the proposition is proved.

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References

[Be] J. Beck, *Braid group action and quantum affine algebras*, Commun. Math. Phys. 165 (1994), 555-568.

[B] S. Berman, *On generators and relations for certain involutory subalgebras Kac-Moody Lie algebras*, Commun. Alg. 17(12) (1989), 3165-3185.

[BM] S. Berman and R. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Inv. Math. (1992), 323-347.

[C] I. Cherednik, *Double-affine Hecke algebras, Knizhink-Zamolodchikov equations and Macdonald's operators*, Intern. Math. Research Notices 9 (1992), 171-180.

[CP] V. Chari and A. Pressly, *Quantum affine algebras and affine Hecke algebras*, Pacific Jour. of Math. 174 No.2 (1996), 295-326.

[D1] V. Drinfeld, *Quantum groups*, Proc. ICM Berkeley 1 (1986), 789-820.

[D2] V. Drinfeld, *A new realization of Yangians and quantized affine algebras*, Sov. Math. Dokl. 36 (1988), 212-216.

[EM] S. Eswara Rao and R. V. Moody, *Vertex representations for N-toroidal Lie algebras and a Generalization of the Virasoro algebra*, Commun. Math. Phys. 159 (1994), 239-264.

[FJ] I. Frenkel and N. Jing, *Vertex representations of quantum affine algebras*, Proc. Natl. Acad. Sci. USA 85 (1988), 9373-9377.

[FK] I. Frenkel and V. Kac, *Basic Representations of affine Lie algebras and dual Resonance models*, Inv. Math. 62 (1980), 23-66.

[GKV] V. Ginzburg, M. Kapranov and E. Vasserot, *Langlands reciprocity for algebraic surface*, Math. Res. Lrtt. 2 (1995), 147-160.

[K] Y. Koyama, *Staggered polarization of vertex model with $U_q(\widehat{\mathfrak{sl}}(n))$-symmetry*, Commun. Math. Phys. 164 (1994), 277-291.

[MEY] R. V. Moody, S. Eswara Rao and T .Yokomuma, *Toroidal Lie algebras and vertex representations*, Geom. Dedicata 35 (1990), 283-307.

[S] P. Slodowy, *Beyond Kac-Moody algebras, and inside*, Canadian Math. Soc. Conference Proceedings Vol.5 (1986), 361-371.

[V] E. Vasserot, *Private communication.*

[VV] M. Varagnolo and E. Vasserot, *Schur duality in the toroidal setting* Commun. Math. Phys. 182 (1996), 469-484.

[Y] H. Yamada, *Extended affine Lie algebra and their vertex representations*, Publ. RIMS. 25 (1989), 587-603.

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